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1999 J. Phys. A: Math. Gen. 32 4441

(http://iopscience.iop.org/0305-4470/32/24/309)

J. Phys. A: Math. Gen. **32** (1999) 4441–4456. Printed in the UK PII: S0305-4470(99)00111-0

A new type of duality symmetry in the theory of N-form abelian fields

R Banerjee and B Chakraborty

S N Bose National Centre for Basic Sciences, Block-JD, Sector-III, Salt Lake, Calcutta-700091, India

E-mail: `rabin@boson.bose.res.in` (R Banerjee) and `biswajit@boson.bose.res.in` (B Chakraborty)

Received 11 December 1998

Abstract. A new type of duality symmetry is introduced in the momentum space, by Fourier transforming a system of N-form abelian free fields. Exploiting the complex nature of the mode amplitudes, it is shown that the corresponding duality group is both Z_2 and $SO(2)$ for all even dimensions. The connection with the conventional duality symmetry where the group is Z_2 ($SO(2)$) for $D = 4k + 2(4k)$ spacetime dimensions is discussed in detail.

1. Introduction

The conventional interpretation of duality symmetry, as briefly reviewed below, leads to distinct $Z_2(SO(2))$ groups in $4k + 2(4k)$ dimensions. This shows that there is a basic difference in the duality transformations, which is among the potentials, in these dimensions. In this paper, we introduce a new type of duality symmetry in the theory of N-form abelian free fields defined in any even dimensions. The duality transformations are not among the potentials; rather these are among the mode amplitudes obtained after Fourier decomposing the basic potentials. Taking advantage of the complex nature of the mode amplitudes, it is possible to invoke a duality symmetry that manifests both the Z_2 as well as the $SO(2)$ groups, irrespective of the dimensionality of spacetime.

Historically (for recent reviews see [1]), the source free Maxwell's equations were the first to display the property of duality symmetry which involves a formal $SO(2)$ rotation, apart from a trivial scale factor, in the space of electric and magnetic fields,

$$\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{B} \end{pmatrix} \rightarrow \begin{pmatrix} \boldsymbol{E}' \\ \boldsymbol{B}' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{B} \end{pmatrix} \tag{1.1a}$$

or, equivalently, a $U(1)$ transformation for the combination $(\boldsymbol{E} + \mathrm{i}\boldsymbol{B})$

$$(\boldsymbol{E} + \mathrm{i}\boldsymbol{B}) \rightarrow (\boldsymbol{E}' + \mathrm{i}\boldsymbol{B}') = \mathrm{e}^{-\mathrm{i}\theta}(\boldsymbol{E} + \mathrm{i}\boldsymbol{B}). \tag{1.1b}$$

Using the language of differential 2-forms F and its dual $\tilde{F}$, defined as

$$\begin{aligned} F &= E_i\, \mathrm{d}x^0 \wedge \mathrm{d}x^i + \tfrac{1}{2} F_{ij}\, \mathrm{d}x^i \wedge \mathrm{d}x^j \\ \tilde{F} &= -B_i\, \mathrm{d}x^0 \wedge \mathrm{d}x^i + \tfrac{1}{2} \tilde{F}_{ij}\, \mathrm{d}x^i \wedge \mathrm{d}x^j \end{aligned} \tag{1.2}$$

with $B_i = \frac{1}{2}\epsilon^{ijk}F_{jk}$ and $E_i = F_{0i} = \frac{1}{2}\epsilon^{ijk}\tilde{F}_{jk}$ being the components of the magnetic and electric fields respectively, (1.1*a*) is recast as

$$\begin{pmatrix} F \\ \tilde{F} \end{pmatrix} \to \begin{pmatrix} F' \\ \tilde{F}' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} F \\ \tilde{F} \end{pmatrix}. \tag{1.3}$$

As is well known this is a symmetry of the equations of motion $\mathrm{d}F = 0$ and $\mathrm{d}\tilde{F} = 0$ only, but not of the action $S = \int \mathrm{d}^4x\, \mathrm{tr}(FF - \tilde{F}\tilde{F})$. Incidentally, this analysis is generic for any abelian $N = 2k$-form fields in $D = 4k$ dimensions, for integral k.

The corresponding situation in two dimensions (which is generic for $D = 4k + 2$ dimensions, for integral k) has also been studied. In the case of the free massless scalar field (which can be regarded as a zero form potential) in two dimensions, the equations of motion are invariant under $Z_2 \times SO(1, 1)$ transformations, although the action is not. This difference from the four-dimensional example is attributed to the basic identities governing the dual operations

$$\tilde{\tilde{F}} = -F;\ D = 4k \qquad \tilde{\tilde{F}} = F;\ D = 4k + 2. \tag{1.4}$$

To elevate the duality at the level of the action, it was naturally imperative to define the relevant transformations in terms of the basic variables which are the associated potentials rather than the field tensors. This is possible by rewriting the action in terms of two potentials. Incidentally, the introduction of a second potential $\tilde{A}$ is essentially tied to the fact that the dual field $\tilde{F}$ is closed by the equation of motion, so that one can write $\tilde{F} = \mathrm{d}\tilde{A}$ as an on-shell relation. It was also shown that the duality groups G preserving the invariance of the action were the subgroups of those found earlier that preserve the invariance of the equations of motion. In fact the former was obtained by taking an intersection with $O(2)$, the group of invariance of the energy–momentum tensor ($T_{\mu\nu} \sim (F_\mu F_\nu + F_\nu F_\mu)$) (here the unwritten indices have been summed over). Specifically, these were [2, 3]

$$G = SO(2);\ D = 4k \qquad G = Z_2;\ D = (4k + 2). \tag{1.5}$$

It is clear, therefore, that a fundamental difference is observed in the study of duality symmetry in $4k$ and $4k + 2$ dimensions.

To put the above discussion in a proper perspective, it might be useful to mention that the original study of duality symmetry in the context of the equations of motion can be understood in an alternative way that does not involve these equations at all. Indeed, it is simple to check that by only demanding the invariance of the dual operation $F \to \tilde{F}$ under some transformation (like (1.3)) yields the $SO(2)$ group for four dimensions. Consider, for instance, the following transformation

$$\begin{pmatrix} F \\ \tilde{F} \end{pmatrix} \to \begin{pmatrix} F' \\ \tilde{F}' \end{pmatrix} = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} F \\ \tilde{F} \end{pmatrix}. \tag{1.6}$$

If we demand that the duality condition is preserved under this transformation, i.e. $\tilde{F}'$ is indeed the dual of F', then it follows that $p = s$ and $q = -r$. Hence, up to a trivial scale factor, the transformation matrix in (1.6) can easily be identified with the standard $SO(2)$ matrix (1.3). The same logic also holds for two dimensions where the relevant group is found to be $Z_2 \times SO(1, 1)$ instead of $SO(2)$. Hence the study of duality symmetry truly becomes meaningful only with regard to the respective actions.

The point to be emphasised here is that this duality symmetry refers to the ordinary space duality. The duality is also valid in the corresponding Fourier-transformed quantities. By this we mean that (1.6) can be expressed (for four-dimensional (4D) Maxwell theory for example) in terms of potential one forms $A(x) = A_\mu\, \mathrm{d}x^\mu$ and $\tilde{A}(x) = \tilde{A}_\mu\, \mathrm{d}x^\mu$ as

$$\begin{pmatrix} A(x) \\ \tilde{A}(x) \end{pmatrix} \to \begin{pmatrix} A'(x) \\ \tilde{A}'(x) \end{pmatrix} = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} A(x) \\ \tilde{A}(x) \end{pmatrix} \tag{1.7}$$

which in turn can be written as

$$\begin{pmatrix} A_\mu(k) \\ \tilde{A}_\mu(k) \end{pmatrix} \rightarrow \begin{pmatrix} A'_\mu(k) \\ \tilde{A}'_\mu(k) \end{pmatrix} = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} A_\mu(k) \\ \tilde{A}_\mu(k) \end{pmatrix} \tag{1.8}$$

where $A_\mu(k)$ is obtained from

$$A_\mu(\boldsymbol{x}, t) = \frac{1}{\sqrt{V}} \sum_k \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} A_\mu(\boldsymbol{k}, t) \tag{1.9}$$

by an inverse (spatial) Fourier transformation. Similar relations also hold for $\tilde{A}_\mu(k)$. Incidentally, it turns out that $\boldsymbol{A}(k)$ and $\tilde{\boldsymbol{A}}(k)$ refer to the two polarization states of the electromagnetic potential.

The purpose of the paper is to show that, apart from the above duality symmetry (1.8), one can introduce a different duality symmetry in momentum space, making use of the fact that the Maxwell field is equivalent to an assembly of an infinite number of decoupled (complex) harmonic oscillators (HO). This duality symmetry is therefore not among the potentials; rather it involves the mode amplitudes. Besides, the study of duality symmetry at the level of individual modes is interesting in its own right, as we know that the particle content of such free-field theories are identified with the corresponding excitations in various modes. Incidentally, the duality group for the complex harmonic oscillator† which is not covered in the literature, can be either SO(2) or Z_2 transformation in an enlarged configuration space, labelled in the appropriate manner. As each mode in the Maxwell field represents an HO, the same analysis of HO can be carried out for each mode of the Maxwell field. We can thus show that in the momentum space the Maxwell field displays both $SO(2)$ and Z_2 symmetries. The same holds for other N-form abelian fields in appropriate dimensions.

A comparison with duality symmetry discussed here with the conventional one has been done for $D = 4$ Maxwell theory. It is shown that the duality generators for the $SO(2)$ transformations in the two cases are distinct. The exact relationship between the two infinitesimal duality symmetry transformations has been worked out.

The paper is split into five sections. In section 2, duality symmetry in the complex HO is analysed. Sections 3 and 4 describe the corresponding analysis for the scalar and Maxwell theory, including the comparison with the HO formulation. In section 5 we discuss duality symmetry in Kalb–Ramond fields. Finally some concluding remarks are made in section 6.

2. Duality in the complex harmonic oscillator

Let us next consider the example of the 'complex' HO. They occur naturally as the Fourier modes of several free-field theories and thus will be useful for the subsequent analysis. Besides, this is an instructive example where distinct variable redefinitions are possible which show a reversal of roles of the duality transformations. Consider, therefore, the following Lagrangian

$$L = \tfrac{1}{2}(\dot{\phi}^*\dot{\phi} - \omega^2\phi^*\phi). \tag{2.1}$$

Linearizing the above Lagrangian, by introducing additional variables π and π^* in an enlarged configuration space, one gets

$$L = \tfrac{1}{2}\omega(\pi^*\dot{\phi} + \pi\dot{\phi}^*) - \tfrac{1}{2}\omega^2(\pi^*\pi + \phi^*\phi). \tag{2.2}$$

Labelling $\phi = q_1$ and $\pi = q_2$ and then again in the reverse order i.e. $\phi = q_2$ and $\pi = q_1$, the following 'chiral' forms of the Lagrangian are obtained,

$$L_\pm(Q) = \pm\tfrac{1}{2}\omega Q^\dagger\epsilon\dot{Q} - \tfrac{1}{2}\omega^2 Q^\dagger Q \tag{2.3}$$

with $Q = \binom{q_1}{q_2}$.

† The real case has been analysed in [4–6].

The occurrence of the ϵ matrix indicates that the Lagrangians $L_\pm$ are invariant under the $SO(2)$ transformation $Q \to R^+Q$. Similarly, the improper rotations R^- induce a swapping $L_+ \leftrightarrow L_-$. Using Noether's prescription, the conserved charge for the $SO(2)$ symmetry is found to be

$$G = -\tfrac{1}{2}\omega Q^\dagger Q. \tag{2.4}$$

To show that this indeed generates the infinitesimal duality transformation

$$\delta q_\alpha = \theta\{q_\alpha, G\} = \theta\epsilon_{\alpha\beta}q_\beta \tag{2.5}$$

recourse is taken to the fundamental brackets

$$\{q_\alpha, q_\beta^*\} = -\frac{2}{\omega}\epsilon_{\alpha\beta} \tag{2.6}$$

following from the symplectic structure of the first-order Lagrangian (2.3). Now consider the following alternative way of relabelling the (ϕ, π) variables in (2.2)

$$\phi = q_1 : \pi = \mathrm{i}q_2 \tag{2.7a}$$

and then as

$$\phi = q_2 : \pi = -\mathrm{i}q_1 \tag{2.7b}$$

which yield the following structures for the Lagrangians,

$$L_\pm(Q) = \tfrac{1}{2}(\pm\mathrm{i}\omega\dot{Q}^\dagger\sigma^1 Q - \omega^2 Q^\dagger Q) \tag{2.8}$$

where $\sigma^1 = \left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)$ is the first Pauli matrix. These Lagrangians are invariant under a discrete Z_2 transformation $Q \to \sigma^1 Q$, while the swapping $L_+ \leftrightarrow L_-$ is effected by $Q \to \epsilon Q$. Clearly therefore, the roles of proper and improper rotations are reversed from the previous case. Compared to the real example [4, 6], the complex HO has a richer symmetry structure that is essentially tied to the complex nature of the variables, allowing for alternative redefinitions.

To complete the analysis, the soldering of the complex 'chiral' oscillators (2.3) is done to reproduce the complex HO. The soldering mechanism, suggested in [8] and extensively developed in [7, 9, 10], is a means of obtaining fresh equivalences by starting from two distinct theories displaying the opposite aspects of some symmetry-like chirality, duality, etc. In the present context, we start from $L_+(Q)$ and $L_-(R)$, regarded as functions of distinct variables Q and R, respectively. Consider a transformation

$$W \to W' = W + \eta. \tag{2.9}$$

Under this, (2.3) transforms as

$$\delta L_\pm(W) = \eta^\dagger\epsilon J^\pm - J^{\pm^\dagger}\epsilon\eta; \;\; W = (Q, R) \tag{2.10}$$

where,

$$J^\pm(W) = \pm\tfrac{1}{2}\omega\dot{W} + \tfrac{1}{2}\omega^2\epsilon W. \tag{2.11}$$

Introduce a (column) matrix-valued variable B transforming as

$$\delta B = -\epsilon\eta. \tag{2.12}$$

Then the first iterated Lagrangian $L_\pm^{(1)}$, defined as

$$L_\pm^{(1)} = L_\pm - B^\dagger J^\pm - (J^\pm)^\dagger B \tag{2.13}$$

can be shown to transform as

$$\delta L_\pm^{(1)} = \mp\tfrac{1}{2}\omega(B^\dagger\dot{\eta} + \dot{\eta}^\dagger B) - \tfrac{1}{2}\omega^2(B^\dagger\epsilon\eta - \eta^\dagger\epsilon B). \tag{2.14}$$

Assuming that B does not depend on W, one finds

$$\delta L_+^{(1)}(Q) + \delta L_-^{(1)}(R) = -\omega^2(B^\dagger\epsilon\eta_- \eta^\dagger\epsilon B). \tag{2.15}$$

Now the soldered Lagrangian L^s is defined as

$$L^s = L_+^{(1)}(Q) + L_-^{(1)}(R) - \omega^2 B^\dagger B. \tag{2.16}$$

Using (2.12) and (2.15), one can easily show that L^s is invariant

$$\delta L^s = 0 \tag{2.17}$$

under the above transformations (2.9) and (2.12). Eliminating the auxilliary variables B and $B^\dagger$ from (2.16), by using the corresponding equations of motion, one obtains

$$L^s = \tfrac{1}{2}(\dot{S}^\dagger\dot{S} - \omega^2 S^\dagger S) \tag{2.18}$$

where

$$S = \tfrac{1}{\sqrt{2}}(Q - R) \tag{2.19}$$

is a 'gauge invariant' combination of variables. Thus starting with chiral forms of Lagrangians $L_+(Q)$ and $L_-(R)$, given as functions of Q and R respectively, we have constructed a soldered Lagrangian L^s, which is a function of the difference S (2.19) only. Thus the bi-dimensional complex HO is manifestly invariant under the simultaneous transformation, $\delta Q = \delta R = \eta$. Note that we have a $U(2)$ symmetry $S \to US$ (here U is a $U(2)$ matrix) in (2.18) in contrast to the $O(2)$ symmetry found for the real HO. Once again, this manifests a richer structure of the complex HO. Here we would like to mention that the same form of soldered Lagrangian (2.18) is obtained if we solder the Lagrangians in (2.8). Using these concepts, the duality symmetry in the context of free-field theories is better understood, as evolved in the subsequent sections.

3. Massless scalar fields in 1 + 1 dimensions

The HO is quite ubiquitious in field theoretical models. This is because a large number of free-field models can be thought of as an assembly of an infinite number of free HOs, each designated by the mode vector $\boldsymbol{k}$. In this section, we shall carry out the mode analysis of the massless scalar fields in the $(1 + 1)$ dimension and study the duality symmetry through these modes, simultaneously revealing the close connection with the HO analysis carried out in the previous section.

The Lagrangian of the model is given by

$$L = \frac{1}{2}\int \mathrm{d}x\, (\dot{\phi}^2(x) - \phi'^2(x)). \tag{3.1}$$

Putting the system in a box of length L, one can make the Fourier decomposition of the real scalar field $\phi(x)$ as

$$\phi(x) = \frac{1}{\sqrt{L}}\sum_k \mathrm{e}^{\mathrm{i}kx}\phi_k(t). \tag{3.2}$$

Here k represents the space component of a 2-vector k^μ, satisfying $k^\mu k_\mu = \omega_k^2 - k^2 = 0$ and $\phi_k^* = \phi_{-k}$. Substituting (3.2) in (3.1), one gets

$$L = \sum_k L_k \tag{3.3a}$$

with

$$L_k = \tfrac{1}{2}(\dot{\phi}_k^*\dot{\phi}_k - \omega_k^2\phi_k^*\phi_k) \tag{3.3b}$$

representing a 'complex' HO for the kth mode (see (2.1)), as ϕ_k is a complex number in general. Thus one can proceed just as in the preceding section to linearize the Lagrangian and then relabel the variables in the appropriate manner to obtain the following duality invariant forms of the Lagrangian,

$$L_{k\pm}(Q_k) = \pm\tfrac{1}{2}\omega_k Q_k^\dagger \epsilon \dot{Q}_k - \tfrac{1}{2}\omega_k^2 Q_k^\dagger Q_k \tag{3.4a}$$

and

$$L_{k\pm}(Q_k) = \tfrac{1}{2}(\pm \mathrm{i}\omega_k \dot{Q}_k^\dagger \sigma^1 Q_k - \omega_k^2 Q_k^\dagger Q_k) \tag{3.4b}$$

with $Q_k = \binom{q_{1k}}{q_{2k}}$. Note that these expressions are just (2.3) and (2.8), but with only an additional subscript kth mode index. It is clear that while the duality group is $SO(2)$ for (3.4a), it is Z_2 for (3.4b). Recall that, expressed in terms of the original scalar fields, only the latter is manifested [2, 3, 7].

We can now proceed with the soldering of these two Lagrangians $L_{k+}(Q)$ and $L_{k-}(R)$, for two independent variables Q and R, as we have done in the previous section to finally get

$$L_k^s = \tfrac{1}{2}(\dot{S}_k^\dagger \dot{S}_k - \omega_k^2 S_k^\dagger S_k) \tag{3.5a}$$

where

$$S_k = \tfrac{1}{\sqrt{2}}(Q_k - R_k) \tag{3.5b}$$

is the 'gauge invariant' combination of variables Q_k and R_k. Note that the above result follows irrespective of whether one starts from (3.4a) or (3.4b). The soldered Lagrangian, which is just the expression for the kth mode, is thus manifestly invariant under the simultaneous transformation, $\delta Q_k = \delta R_k = \eta_k$. At this stage we can sum over all the modes to get the complete soldered Lagrangian L^s as

$$L^s = \sum_k L_k^s = \frac{1}{2}\sum_k (\dot{S}_k^\dagger \dot{S}_k - \omega_k^2 S_k^\dagger S_k). \tag{3.6}$$

Using the inverse Fourier transform, this can be easily shown to yield

$$L^s = \frac{1}{2}\int \mathrm{d}x\, \partial_\mu S^\dagger(x)\partial^\mu S(x) \tag{3.7}$$

where

$$S(x) = \frac{1}{\sqrt{L}}\sum_k \mathrm{e}^{\mathrm{i}kx} S_k(t) \tag{3.8}$$

is a doublet of real scalar fields. This is again given in terms of the difference

$$S(x) = \frac{1}{\sqrt{2}}(Q(x) - R(x)) \tag{3.9}$$

where $Q(x)$ and $R(x)$ are obtained from Q_k and R_k using expressions similar to (3.8).

On the other hand, as shown in [7], the original model (3.1) can be re-expressed, after a suitable redefinition of variables, in a linearized form as

$$\mathcal{L}_\pm(\Phi) = \tfrac{1}{2}(\pm\dot{\Phi}^T \sigma^1 \Phi' - \Phi'^T \Phi') \tag{3.10}$$

where $\Phi = \binom{\phi_1}{\phi_2}$. The matrix swapping $\mathcal{L}_+ \leftrightarrow \mathcal{L}_-$ is ϵ. Again as shown in [7], the soldering of $\mathcal{L}_+(Q)$ and $\mathcal{L}_-(R)$, where Q and R denote the independent fields corresponding to the positive and negative components of the Lagrangian given in (3.10) yields

$$\mathcal{L}^s = \tfrac{1}{2}\partial_\mu S^\dagger \partial^\mu S \tag{3.11}$$

where S is identical to (3.9). Note that this is precisely the Lagrangian density appearing in (3.7). This shows that writing the original model in the chiral form and then soldering, as in

[7], yields the same result as that obtained by first making a Fourier decomposition (3.2) and then expressing this Lagrangian (a 'complex HO') in a linearized chiral oscillator form $L_{k\pm}$ (3.4), next soldering to get L_k^s (3.5), followed by a final summation over all the modes to get (3.7). Schematically, this can be represented as†

$$\begin{array}{ccccc} L \rightarrow & L_{\pm} & \rightarrow L^s \\ \downarrow & \uparrow & \uparrow \\ L_k \rightarrow & L_{k\pm} & \rightarrow L_k^s. \end{array}$$

It may be recalled that (3.10) is the conventional form of the duality symmetric action in two dimensions [2, 3, 7]. Nevertheless, expressed in terms of its modes, the massless scalar theory (3.3*b*) gets mapped to the complex HO, thereby manifesting either the Z_2 or the $SO(2)$ symmetry depending on the variable redefinitions. To establish compatibility with (3.10) where only the Z_2 symmetry is revealed, recall that (3.10) was obtained [4] by rewriting (3.1) in its linearized version

$$\mathcal{L} = \tfrac{1}{2}(P\dot{\phi} - \dot{P}\phi - P^2 - \phi'^2) \qquad (3.12)$$

where P is an additional variable in an extended configuration space. In order to get the form $\mathcal{L}_+$ (3.10) for example, one has to make the following relabelling

$$\phi = \phi_1 \qquad (3.13a)$$

$$P = \phi_2'. \qquad (3.13b)$$

Incidentally, the existence of the second scalar field $\phi_2(x)$ is understood in the following manner. Since $\phi(x) = \phi_1(x)$ can be regarded as a 0-form potential, the field 1-form

$$F = \mathrm{d}\phi_1 = (\dot{\phi}_1\,\mathrm{d}t + \phi_1'\,\mathrm{d}x)$$

has the dual

$$\tilde{F} = -(\dot{\phi}_1\,\mathrm{d}x + \phi_1'\,\mathrm{d}t)$$

which is closed on-shell, so that in the absence of any non-trivial topology it must be exact too. In other words there exists another function $\phi_2(x)$ satisfying $\tilde{F} = -\mathrm{d}\phi_2$. As can be easily seen, here $P = \dot{\phi}_1 = \phi_2'$.

To get the form $\mathcal{L}_-$, the relabelling has to be done in the reverse order, i.e. ($\phi = \phi_2$: $P = \phi_1'$). In the rest of this section, we shall only consider $\mathcal{L}_+$ for convenience.

At this stage, we Fourier analyse the field $\phi_\alpha(x)$ ($\alpha = 1, 2$) and $P(x)$ as

$$\phi_\alpha(x) = \frac{1}{\sqrt{L}}\sum_k \mathrm{e}^{\mathrm{i}kx}\phi_{\alpha k}(t) \qquad (3.14)$$

$$P(x) = \frac{1}{\sqrt{L}}\sum_k \mathrm{e}^{\mathrm{i}kx}k\pi_k(t). \qquad (3.15)$$

We can easily see that (3.13*b*) implies, in terms of momentum space variables,

$$\pi_k = \mathrm{i}\phi_{2k}. \qquad (3.16)$$

Again, reality of $\phi_\alpha(x)$ implies

$$\phi^*_{\alpha k} = \phi_{\alpha(-k)} \qquad (3.17a)$$

$$\pi^*_k = -\pi_{-k}. \qquad (3.17b)$$

One can then proceed, as for the model (3.1), to obtain two equivalent forms for L_k, starting from (3.12). Using the Fourier decomposition of both ϕ and P fields, we get

$$L_k = \tfrac{1}{2}[k(\pi_k\dot{\phi}^*_k + \pi^*_k\dot{\phi}_k) - \omega_k^2(\phi^*_k\phi_k + \pi^*_k\pi_k)]. \qquad (3.18)$$

† Note an important distinction between the two rows of the diagram. While the first manifests only the familiar Z_2 symmetry, the second is associated with both Z_2 and $SO(2)$ symmetries.

Note that in the Fourier decomposition of the field $P(x)$ in (3.15), we had intentionally incorporated an additional factor of k in front of π_k, so that the form of (3.18) looks exactly like that of 'complex' HO (2.2). Also note that the relation $\omega_k^2 = k^2$ was used crucially in these expressions, indicating that the above structure for the Lagrangian is strictly valid for massless fields.

Mimicing the steps of the complex HO, it is simple to show that the above Lagrangian displays either the Z_2 or $SO(2)$ symmetry, based on a suitable relabelling of fields. The generator for the latter can be obtained simply by generalizing (2.4) to get

$$G_k = -\tfrac{1}{2}\omega_k Q_k^\dagger Q_k \tag{3.19}$$

where Q_k is the doublet $\binom{\phi_k}{\pi_k}$.

At this stage, we would like to make an observation. The conventional duality symmetry for $\mathcal{L}_\pm$ is given by the discrete group Z_2 acting on the space of doublet of potentials $\binom{\phi_1(x)}{\phi_2(x)}$ or equivalently in the Fourier transformed doublet $\binom{\phi_{1k}}{\phi_{2k}}$, i.e. in the momentum space. On the other hand, the duality group we have introduced acts on the doublet $\binom{q_{1k}}{q_{2k}} = \binom{\phi_k}{\pi_k} = \binom{\phi_{1k}}{\mathrm{i}\phi_{2k}}$ and is therefore distinct from the conventional one. Some general remarks are now in order distinguishing more clearly the (momentum space) duality symmetry given here and the conventional (coordinate space) duality symmetry. Consider, for example, massive scalar fields in D-dimensional spacetime,

$$L = \int \mathrm{d}^{D-1}x\, \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2). \tag{3.20}$$

It can now be easily seen that only the massless case in $D = 2$ admits a duality invariant chiral form of the Lagrangian (3.10). For all other dimensions D, the above Lagrangian (3.20) (massive or massless) does not admit a duality invariant chiral form. In these cases, the scalar field $\phi(x)$ can no longer be regarded as a 0-form potential and the correspondence with the theory of N-form abelian fields breaks down. Consequently a (conventional) duality invariant chiral form of L, that would be a counterpart of (3.10), cannot be written down. On the contrary, this model too can be re-expressed, upon Fourier analysis, just as in (3.3), with

$$\omega_k^2 = \boldsymbol{k}^2 + m^2. \tag{3.21}$$

It can then be linearized and relabelled appropriately to get similar expressions as that of (3.4*a*) and (3.4*b*), exhibiting $SO(2)$ and Z_2 symmetries, respectively. Note that this can be done for any D, including odd D. Thus the duality transformations in the momentum space still exist in these cases, in spite of the absence of the duality symmetry of fields in the configuration space.

4. Maxwell field in 4D

In this section we shall carry out a similar analysis for the free Maxwell field. But because of the inherent gauge invariance of the model, we shall not start with a Fourier analysis right at the beginning. Rather the Gauss constraint of the model will be imposed strongly to isolate the physical degrees of freedom. Mode analysis then reveals the HO structure just as in the scalar case with the difference that to each mode $\boldsymbol{k}$ there are two orthogonal transverse oscillators. Following the HO example, this model is then linearized and written in 'chiral' forms. We then carry out the soldering of the 'chiral' forms of the Lagrangian followed by a summation over all the modes, to get hold of the final soldered Lagrangian. This part is just the same as we did for the scalar field. To that end, consider

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} = \tfrac{1}{2}(\boldsymbol{E}^2 - \boldsymbol{B}^2) \tag{4.1}$$

where $E_i = (\partial_0 A_i - \partial_i A_0)$ and $B^k = \epsilon^{ijk}\partial_i A_j$ are the electric and magnetic fields. At this stage, the Gauss constraint ($\nabla \cdot \boldsymbol{E} = 0$) can be solved for A_0 yielding

$$A_0 = \frac{\partial_0}{\nabla^2}(\nabla \cdot \boldsymbol{A}). \tag{4.2}$$

Time preservation of (4.2) is guaranteed by the equations of motion. Hence it is possible to eliminate A_0 from the Lagrangian by using (4.2) to get [11],

$$\mathcal{L} = \tfrac{1}{2}[(\dot{\boldsymbol{A}}^T)^2 - (\nabla \times \boldsymbol{A}^T)^2] \tag{4.3}$$

where the longitudinal component $\boldsymbol{A}^L$ drops out automatically and only the physical (transverse) component $\boldsymbol{A}^T$ survives. In terms of the gauge field $\boldsymbol{A}$, this is given by

$$A_i^T = P_{ij}A_j = \left(\delta_{ij} - \frac{\partial_i \partial_j}{\boldsymbol{\nabla}^2}\right) A_j \tag{4.4}$$

with P_{ij} being the projection operator satisfying $P^2 = P$. Therefore the Fourier decomposition has to be carried out keeping this in mind. Hereafter, we shall omit the superscript (T) from $\boldsymbol{A}^T$ and write simply $\boldsymbol{A}$.

So finally carrying out a Fourier decomposition

$$\boldsymbol{A}(x) = \frac{1}{\sqrt{V}} \sum \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \boldsymbol{A}_k(t). \tag{4.5}$$

Note that $\boldsymbol{A}_k(t)$ can be written as

$$\boldsymbol{A}_k(t) = \sum_{\lambda=1}^{2} A_{k\lambda}(t)\boldsymbol{\epsilon}_\lambda(k) \tag{4.6}$$

where $\epsilon_\lambda(k)$ are the polarization vectors orthogonal to $\boldsymbol{k}$ ($\boldsymbol{k} \cdot \epsilon_\lambda(k) = 0$). This orthogonality projects out the transverse component of the vector potential.

As expected, the Lagrangian $L(= \int \mathrm{d}^3x\, \mathcal{L})$ can then be written as

$$L = \sum L_k \tag{4.7}$$

with

$$L_k = \tfrac{1}{2}(\dot{\boldsymbol{A}}_k^* \cdot \dot{\boldsymbol{A}}_k - \omega_k^2 \boldsymbol{A}_k^* \cdot \boldsymbol{A}_k). \tag{4.8}$$

In comparison with (3.3*b*), we can see that $\boldsymbol{A}_k$ are now 3-vectors in $\boldsymbol{C}^3$ in contrast to ϕ_k, which are just complex scalars.

Thus one can proceed just as in the scalar fields to linearize (4.8) by invoking additional vector-valued variables $\boldsymbol{\Pi}_k$ and its complex conjugates in an enlarged configuration space, to write

$$L_k = \tfrac{1}{2}\omega_k(\boldsymbol{\Pi}_k^* \cdot \dot{\boldsymbol{A}}_k + \boldsymbol{\Pi}_k \cdot \dot{\boldsymbol{A}}_k^*) - \tfrac{1}{2}\omega_k^2(\boldsymbol{\Pi}_k^* \cdot \boldsymbol{\Pi}_k + \boldsymbol{A}_k^* \cdot \boldsymbol{A}_k) \tag{4.9}$$

associated with each mode k. Parametrizing $\boldsymbol{q}_{1k} = \boldsymbol{A}_k$ and $\boldsymbol{q}_{2k} = \boldsymbol{\Pi}_k$ and then in the reverse order, i.e. $\boldsymbol{q}_{2k} = \boldsymbol{A}_k$ and $\boldsymbol{q}_{1k} = \boldsymbol{\Pi}_k$, one gets the following 'chiral' forms of the Lagrangian

$$L_{k\pm}(\boldsymbol{Q}_k) = \pm\tfrac{1}{2}\omega_k \boldsymbol{Q}_k^\dagger \epsilon \cdot \dot{\boldsymbol{Q}}_k - \tfrac{1}{2}\omega_k^2 \boldsymbol{Q}_k^\dagger \cdot \boldsymbol{Q}_k \tag{4.10}$$

where $\boldsymbol{Q}_k$ is the doublet $\binom{q_{1k}}{q_{2k}}$. The above Lagrangian is invariant, mode by mode, under the usual $SO(2)$ transformation $\boldsymbol{Q}_k \to \epsilon \boldsymbol{Q}_k$. Similarly, under the transformation $\boldsymbol{Q}_k \to \sigma^1 \boldsymbol{Q}_k$ the Lagrangians L_{k+} and L_{k-} are swapped into one another.

Alternatively, parametrizing,

$$\boldsymbol{\Phi}_{1k} = \boldsymbol{A}_k \qquad \boldsymbol{\Phi}_{2k} = -\mathrm{i}\boldsymbol{\Pi}_k \tag{4.11}$$

and then in the reverse order, the Lagrangian (4.9) is expressed in the chiral form as

$$L_{k\pm} = \tfrac{1}{2}(\pm \mathrm{i}\omega_k \dot{\mathbf{\Phi}}_k^\dagger(t)\sigma^1 \mathbf{\Phi}_k(t) - \omega_k^2 \mathbf{\Phi}_k^\dagger(t)\mathbf{\Phi}_k(t)) \tag{4.12}$$

which reveals the Z_2 invariance, instead of the usual $SO(2)$. The analogy with the 'complex' HO is now complete. Not surprisingly, therefore, a similar equation (3.18) has also occurred earlier in the case of scalar field. The only additional feature in this case is that $\mathbf{\Phi}_k = \binom{\mathbf{\Phi}_{1k}}{\mathbf{\Phi}_{2k}}$ is now a doublet of vector fields.

It is quite straightforward to solder the two 'chiral' forms of the Lagrangians $L_{k+}(\boldsymbol{Q}_k)$ and $L_{k-}(\boldsymbol{R}_k)$ in the lines of the scalar case to get

$$L_k^s = \tfrac{1}{2}(\dot{\boldsymbol{S}}_k^\dagger \cdot \dot{\boldsymbol{S}}_k - \omega_k^2 \boldsymbol{S}_k^\dagger \cdot \boldsymbol{S}_k) \tag{4.13a}$$

where,

$$\boldsymbol{S}_k = \frac{1}{\sqrt{2}}(\boldsymbol{Q}_k - \boldsymbol{R}_k) \tag{4.13b}$$

is a doublet of *vectors*. Contrast this with (3.5*b*), where S_k stands for a doublet of *scalars*.

Now to obtain the final soldered Lagrangian, we have to sum over all the modes $(L^s = \sum L_k^s)$, as we did for the scalar case (3.6). This yields

$$L^s = -\frac{1}{4}\int \mathrm{d}^3x\, G^\alpha_{\mu\nu} G^{\alpha\mu\nu} \tag{4.14a}$$

where

$$G^\alpha_{\mu\nu} = \partial_\mu A^\alpha_\nu - \partial_\nu A^\alpha_\mu \tag{4.14b}$$

is a doublet of abelian field strengths ($\alpha = 1, 2$). This is the same result as that obtained in [7]. We have thus been able to provide a second derivation of (4.14) by starting from the basic HO example.

Alternatively, introducing a doublet of a divergence free vector field

$$\boldsymbol{S}(x) = \frac{1}{\sqrt{V}}\sum \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \boldsymbol{S}_k(t) \tag{4.15}$$

one can also cast L^s in the pattern of scalar fields (3.7) as

$$L^s = \frac{1}{2}\int \mathrm{d}^3x\, \partial_\mu \boldsymbol{S}(x)\partial^\mu \boldsymbol{S}(x). \tag{4.16}$$

Again, the only difference with (3.7) is that the $S(x)$ appearing there is a doublet of scalar fields, in contrast to the case here, where $\boldsymbol{S}(x)$ is a doublet of vector fields.

The reason that the soldered Lagrangian for electrodynamics can be cast in the form of scalars is rooted to the fact that, at the level of modes, both electrodynamics and scalar theory represent an infinite number of decoupled HOs. The only additional feature of the former is that to each mode $\boldsymbol{k}$, there exist two orthogonal HOs associated with two polarization states.

4.1. Comparison with conventional duality symmetry

One may have observed that the entire discussion of duality symmetry for the Maxwell theory did not invoke the standard form of the Lagrangian [2, 7, 12]

$$\mathcal{L}_\pm = \tfrac{1}{2}(\pm\epsilon_{\alpha\beta}\boldsymbol{E}_\alpha \cdot \boldsymbol{B}_\beta - \boldsymbol{B}_\alpha \cdot \boldsymbol{B}_\alpha) \tag{4.17a}$$

where

$$E_{i\alpha} = \partial_0 A_{i\alpha} - \partial_i A_{0\alpha}$$

and

$$B_{i\alpha} = \epsilon^{ijk}\partial_j A_{k\alpha} \tag{4.17b}$$

represent the electric and magnetic fields in the internal space. This is further simplified to

$$\mathcal{L} = \tfrac{1}{2}(\epsilon_{\alpha\beta}\dot{\boldsymbol{A}}_\alpha \cdot \boldsymbol{B}_\beta - \boldsymbol{B}_\alpha \cdot \boldsymbol{B}_\alpha) \tag{4.18}$$

since the A_0 piece merely contributes a boundary term. The above form of the duality invariant Lagrangian was obtained in various ways [2, 7, 17] and has also been the starting point of several recent investigations [13–16]. It is therefore desirable to establish some sort of connection of our analysis with this structure. Note that we are only considering the positive 'chiral' component of (4.17) here.

Performing a mode analysis of (4.18), we obtain, for the Lagrangian $L(= \int \mathrm{d}^3x\, \mathcal{L})$

$$L = \sum_k L_k \tag{4.19a}$$

where

$$L_k = \tfrac{1}{2}(\epsilon_{\alpha\beta}\dot{\boldsymbol{A}}^*_{\alpha k} \cdot \boldsymbol{B}_{\beta k} - \boldsymbol{B}^*_{\alpha k} \cdot \boldsymbol{B}_{\alpha k}). \tag{4.19b}$$

Using (4.17b) and the fact that only the transverse components of the fields are relevant, one finds the following relations:

$$\boldsymbol{B}_{1k} = \mathrm{i}\boldsymbol{k} \times \boldsymbol{A}_{1k} \qquad \boldsymbol{B}_{2k} = \omega_k \boldsymbol{\Pi}_k \qquad \boldsymbol{A}_{2k} = \frac{\mathrm{i}}{\omega_k}\boldsymbol{k} \times \boldsymbol{\Pi}_k = \frac{\mathrm{i}}{\omega_k^2}\boldsymbol{k} \times \boldsymbol{B}_{2k}. \tag{4.20}$$

Inserting these in (4.19) yields

$$L_k = \tfrac{1}{2}\omega_k(\boldsymbol{\Pi}^*_k \cdot \dot{\boldsymbol{A}}_{1k} + \boldsymbol{\Pi}_k \cdot \dot{\boldsymbol{A}}^*_{1k}) - \tfrac{1}{2}\omega_k^2(\boldsymbol{\Pi}^*_k \cdot \boldsymbol{\Pi}_k + \boldsymbol{A}^*_{1k} \cdot \boldsymbol{A}_{1k}) \tag{4.21}$$

which reproduces (4.9). This shows the equivalence of the duality invariant Maxwell action derived here with the conventional form.

Let us now consider the generators of the duality transformation. First, the conventional duality transformation

$$\delta \boldsymbol{A}_\alpha = \theta \epsilon_{\alpha\beta} \boldsymbol{A}_\beta \tag{4.22}$$

for angle θ, as applied to the Lagrangian (4.18) will be discussed. Note that, we have suppressed the mode index k for the time being. An application of Noether's theorem yields the generator

$$G = \tfrac{1}{2}\boldsymbol{A}^*_\alpha \boldsymbol{B}_\alpha \tag{4.23}$$

which has the desired Chern–Simons form in the momentum space. To check explicitly that G generates the transformation (4.22), use of the symplectic brackets following from a constrained analysis of (4.18),

$$\{A_{\alpha i}, A^*_{\beta j}\} = -2\mathrm{i}\epsilon_{\alpha\beta}\epsilon_{ijl}\frac{k_l}{\boldsymbol{k}^2} \tag{4.24}$$

is made. Using (4.23) and (4.24), it follows that

$$\delta A_{\alpha i} = \theta\{A_{\alpha i}, G\} = \theta \epsilon_{\alpha\beta} A_{\beta i} \tag{4.25}$$

as desired.

Let us next consider the oscillator-like structure (4.10), which may be re-expressed as

$$L = \tfrac{1}{2}\omega \epsilon_{\alpha\beta} q^*_{\alpha i} \dot{q}_{\beta i} - \tfrac{1}{2}\omega^2 q^*_{\alpha i} q_{\alpha i}. \tag{4.26}$$

As discussed in the HO case, this is invariant under the duality transformation,

$$\delta_0 q_{\alpha i} = \theta \epsilon_{\alpha\beta} q_{\beta i} \tag{4.27}$$

where δ_0 has been used instead of δ to indicate that these operations are in distinct spaces. To repeat, δ involves a transformation in the potential $\binom{\boldsymbol{A}_1}{\boldsymbol{A}_2}$ space, while δ_0 involves a transformation in the $\binom{\boldsymbol{A}}{\boldsymbol{\Pi}} = \binom{q_1}{q_2}$ space. We shall subsequently derive an exact relation between δ and δ_0.

The fundamental symplectic structure in this first-order Lagrangian (4.26) can be easily read. This is given by the brackets

$$\{q_{\alpha i}, q^*_{\beta j}\} = -\frac{2}{\omega}\epsilon_{\alpha\beta}\left(\delta_{ij} - \frac{k_i k_j}{\boldsymbol{k}^2}\right). \tag{4.28}$$

Note the similarity with the algebra (2.6) found earlier in the HO case. The only distingushing feature is the presence of the *transverse* Kronecker delta in place of the ordinary Kronecker delta to account for the transversality condition

$$k_i q_{\alpha i} = k_i q^*_{\alpha i} = 0. \tag{4.29}$$

Using the brackets (4.28), it is quite trivial to show that the form of the generator

$$G_0 = -\frac{\omega}{2} q^*_{\alpha i} q_{\alpha i} \tag{4.30}$$

obtained by using Noether's prescription, indeed generates the appropriate transformation (4.27).

To connect δ and δ_0, recall that (4.22) implies

$$\delta \boldsymbol{B}_\alpha = \theta \epsilon_{\alpha\beta} \boldsymbol{B}_\beta \tag{4.31}$$

where $\boldsymbol{B}_\alpha = \mathrm{i}\boldsymbol{k} \times \boldsymbol{A}_\alpha$ in the k-space. From the identifications (4.20), it transpires that

$$\delta\boldsymbol{\Pi} = -\frac{\theta}{\omega}\boldsymbol{B}_1 = -\frac{\mathrm{i}\theta}{\omega}(\boldsymbol{k} \times \boldsymbol{A}_1) \qquad \delta\boldsymbol{A} = \delta\boldsymbol{A}_1 = \theta\boldsymbol{A}_2 = \frac{\mathrm{i}\theta}{\omega}(\boldsymbol{k} \times \boldsymbol{\Pi}). \tag{4.31}$$

Combining the above two equations in (4.31), one gets

$$\delta\begin{pmatrix}\boldsymbol{\Pi}\\ \boldsymbol{A}\end{pmatrix} = -\frac{\mathrm{i}}{\omega}\boldsymbol{k} \times \begin{pmatrix}\theta\boldsymbol{A}\\ -\theta\Pi\end{pmatrix} = -\frac{\mathrm{i}}{\omega}\boldsymbol{k} \times \delta_0\begin{pmatrix}\boldsymbol{\Pi}\\ \boldsymbol{A}\end{pmatrix}. \tag{4.32}$$

From where it follows that

$$\delta = -\frac{\mathrm{i}}{\omega}\boldsymbol{k} \times \delta_0. \tag{4.33}$$

This is the cherished relation between the duality transformations in the two distinct spaces.

5. Kalb–Ramond theory in six dimensions

The action is given by

$$S = \frac{1}{12}\int \mathrm{d}^6 x\, \mathrm{tr}(FF) \tag{5.1}$$

where F is the field 3-form

$$F = \frac{1}{3!}F_{\mu\nu\lambda}\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\lambda. \tag{5.2}$$

In terms of the electric and magnetic fields

$$E_{ij} = F_{0ij} \qquad B_{ij} = \frac{1}{3!}\epsilon^{ijklm}F_{klm} \tag{5.3}$$

the Lagrangian takes the form

$$L = \frac{1}{2}\int \mathrm{d}^5 x\, (E_{\langle ij\rangle}E_{\langle ij\rangle} - B_{\langle ij\rangle}B_{\langle ij\rangle}). \tag{5.4}$$

Here $\langle ij \rangle$ indicates a certain ordering for the spatial indices ij. Let us use only the ascending order. So the summation over the spatial indices in (5.4) has to be carried out with an ascending order of these indices. A canonical analysis leads to the Hamiltonian

$$H = \frac{1}{2}\int \mathrm{d}^5x\,(E_{\langle ij \rangle}E_{\langle ij \rangle} + B_{\langle ij \rangle}B_{\langle ij \rangle} + A_{0i}\partial_j\pi_{ij}) \tag{5.5}$$

where $\pi_{\mu\nu}$ is the momenta canonically conjugate to $A_{\mu\nu}$:

$$\pi_{\langle ij \rangle} = \frac{\delta L}{\delta \dot{A}_{\langle ij \rangle}} = E_{\langle ij \rangle} \qquad \pi_{0i} = \frac{\delta L}{\delta \dot{A}_{0i}} = 0. \tag{5.6}$$

Clearly A_{0i} plays the role of a Lagrange multiplier enforcing the Gauss constraint

$$G_i = \partial_j\pi_{ij} \approx 0. \tag{5.7}$$

This Gauss constraint subjected to a Coulomb-like gauge

$$\partial_j A_{ij} = 0 \tag{5.8}$$

simplifies to,

$$\nabla^2 A_{0i} = \partial_i\partial_j A_{0j}. \tag{5.9}$$

A general solution of (5.9) is

$$A_{0i} = \partial_i f \tag{5.10}$$

where $f(x)$ is an arbitrary differentiable function. Exploiting the freedom in the time-independent gauge transformation on A_{0i}, the solution of (5.10) is gauge equivalent to the trivial solution

$$A_{0i} = 0. \tag{5.11}$$

The Lagrangian (5.4), therefore reduces to

$$L = \frac{1}{2}\int \mathrm{d}^5x\,(\dot{A}^2_{\langle ij \rangle} - B^2_{\langle ij \rangle}). \tag{5.12}$$

As before, the above Lagrangian is linearized by introducing auxilliary variables

$$L = \frac{1}{2}\int \mathrm{d}^5x\,([P_{\langle ij \rangle}\dot{A}_{\langle ij \rangle} - \dot{P}_{\langle ij \rangle}A_{\langle ij \rangle}] - [P^2_{\langle ij \rangle} + B^2_{\langle ij \rangle}]). \tag{5.13}$$

Again, as before, we can introduce dual ($\tilde{F}$) of the field (F) 3-form (5.2) and express P_{ij} in terms of this $\tilde{F}$ as

$$P_{ij} = \frac{1}{3!}\epsilon^{ijklm}\tilde{F}_{klm}. \tag{5.14}$$

On the other hand, the equation of motion implies

$$\mathrm{d}\tilde{F} = 0. \tag{5.15}$$

Again this closure of $\tilde{F}$ allows one to introduce another 2-form potential $\tilde{A}$

$$\tilde{F} = \mathrm{d}\tilde{A} \tag{5.16}$$

as an on-shell relation. Next, introducing a renaming of variables

$$A_{ij} = A^{(1)}_{ij} \tag{5.17a}$$

$$\tilde{A}_{ij} = A^{(2)}_{ij} \tag{5.17b}$$

$$B_{ij} = B^{(1)}_{ij} \tag{5.17c}$$

$$P_{ij} = B^{(2)}_{ij} \tag{5.17d}$$

we obtain the desired form of the Lagrangian

$$L = \frac{1}{2}\int d^5x\, [B^{(\alpha)}_{\langle ij\rangle}\sigma^1_{\alpha\beta}\dot{A}^{(\beta)}_{\langle ij\rangle} - B^{(\alpha)}_{\langle ij\rangle}B^{(\alpha)}_{\langle ij\rangle}] \tag{5.18}$$

which obviously has the desired Z_2 symmetry.

After exposing the conventional duality symmetry in the space of potentials, we now consider the alternative form obtained by first Fourier decomposing A_{ij},

$$A_{ij}(\boldsymbol{x}, t) = \frac{1}{\sqrt{V}}\sum_k e^{i\boldsymbol{k}\cdot\boldsymbol{x}} A_{ij}(\boldsymbol{k}, t). \tag{5.19}$$

Imposing the transversality (gauge) condition

$$C_i = k_j A_{ij}(k) = 0 \tag{5.20}$$

it is found that the original Lagrangian (5.4) may be recast as

$$L = \frac{1}{2}\sum_k [\dot{A}^*_{\langle ij\rangle}(k)A_{\langle ij\rangle}(k) - \omega_k^2 A^*_{\langle ij\rangle}(k)A^*_{\langle ij\rangle}(k)]. \tag{5.21}$$

This clearly represents an assembly of HO, constrained by (5.20). Like Maxwell's case this too is a reducible system, as $k_i C_i = 0$. As before, by suitable reparametrizations, both Z_2 and $SO(2)$ duality symmetries can be manifested from (5.21). Consider the $SO(2)$ case first. The form of the Lagrangian (5.21) can be linearized as

$$L_k = \tfrac{1}{2}\omega_k(\Pi^*_{\langle ij\rangle}(k)\cdot\dot{A}_{\langle ij\rangle}(k) + \Pi_{\langle ij\rangle}(k)\cdot\dot{A}_{\langle ij\rangle}(k)^*) \\ -\tfrac{1}{2}\omega_k^2(\Pi^*_{\langle ij\rangle}(k)\cdot\Pi_{\langle ij\rangle}(k) + A^*_{\langle ij\rangle}(k)\cdot A_{\langle ij\rangle}(k)). \tag{5.22}$$

Then relabelling $q_{1\langle ij\rangle} = \pi_{\langle ij\rangle}$ and $q_{2\langle ij\rangle} = A_{\langle ij\rangle}$ and then again in the reverse order, one gets the following chiral forms of the Lagrangian

$$L_{k\pm}(\boldsymbol{Q}) = \pm\tfrac{1}{2}\omega_k \boldsymbol{Q}^\dagger_{\langle ij\rangle}\epsilon\cdot\dot{\boldsymbol{Q}}_{\langle ij\rangle} - \tfrac{1}{2}\omega_k^2\boldsymbol{Q}^\dagger_{\langle ij\rangle}\cdot\boldsymbol{Q}_{\langle ij\rangle} \tag{5.23}$$

where $\boldsymbol{Q}_{\langle ij\rangle} = \binom{q_{1\langle ij\rangle}}{q_{2\langle ij\rangle}}$. Clearly this has the requisite $SO(2)$ symmetry. The corresponding generator, obtained by using Noether's prescription, is given by

$$G = -\frac{\omega}{2}\boldsymbol{Q}^\dagger_{\langle ij\rangle}\boldsymbol{Q}_{\langle ij\rangle}. \tag{5.24}$$

To reveal the Z_2 symmetry, one has to parametrize

$$\Phi_{1\langle ij\rangle} = A_{\langle ij\rangle} \tag{5.25a}$$

along with

$$\Phi_{2\langle ij\rangle} = -i\Pi_{\langle ij\rangle} \tag{5.25b}$$

and then in the reverse order. With this the above expression of the Lagrangian (5.21) takes the form

$$L_{k\pm} = \tfrac{1}{2}(\pm i\omega_k\dot{\Phi}^\dagger_{\langle ij\rangle}\sigma^1\Phi_{\langle ij\rangle} - \omega_k^2\Phi^\dagger_{\langle ij\rangle}\Phi_{\langle ij\rangle}) \tag{5.26}$$

with $\Phi_{\langle ij\rangle} = \binom{\Phi_{1\langle ij\rangle}}{\Phi_{2\langle ij\rangle}}$. This clearly has the requisite Z_2 symmetry, rather than $SO(2)$ symmetry.

A generalization to higher-order abelian N-form fields is quite straightforward.

6. Conclusions

This paper showed that duality symmetry in certain free-field theories had their origin in a similar symmetry in a quantum mechanical example—the 'complex' harmonic oscillator (HO). Our analysis has revealed that the study of duality symmetries in the HO case is fundamental to properly understand the corresponding phenemenon in the field-theoretic case, at least for the models considered in this paper. Indeed by performing an explicit mode analysis, the free scalar, Maxwell and Kalb–Ramond theories were mapped to the complex HO. The one-to-one correspondence between duality symmetry in the HO and the field theories was easily established.

An algebraic consistency check was also provided for the mode analysis. This was done by taking recourse to the soldering mechanism that was earlier advocated by one of us [7]. It was shown that the soldering of duality symmetric Lagrangians ($\mathcal{L}_+$ and $\mathcal{L}_-$) before the mode decomposition yields identical results by, alternately, first doing a mode analysis of the individual Lagrangians, then soldering the various modes and finally summing over all the modes. This has been depicted pictorially in a figure.

To understand the new feature in this paper it is necessary to recall the development of duality symmetry. Originally, by considering the transformations on the electric and magnetic fields, an invariance of the equations of motion was found although the Lagrangian flipped its sign. In fact, as shown here, this duality is obtained directly from the algebraic transformation theory and need not consider any equations of motion. Obviously, therefore, it was necessary to look at the invariance of the action. Moreover, since the electric and magnetic fields are derived quantities from the potential, it was reasonable to study duality symmetry through these potentials. Simultaneously this brought out a new feature, namely, the invariance of the action itself. Nevertheless, a distinction between twice-odd and twice-even dimensions prevailed since the duality groups in the two cases differed. By pushing this development to its logical conclusion of considering the potential not as a basic field, but as a quantity derived from its Fourier modes, and then investigating duality symmetry through these modes, we obtained new results. The invariance of the action has been demonstrated for both the duality groups Z_2 and $SO(2)$, irrespective of the dimensionality of space time. By suitable variable redefinitions it was possible to discuss the role of either Z_2 or $SO(2)$ as a duality group in all the models. The germ of this feature was obviously contained in the HO, which displayed both the symmetries depending on the change of variables. It should be stressed that the duality symmetry discussed here is among the mode amplitudes and hence distinct from the usual analysis which considers the potentials. Indeed the explicit relation between the infinitesimal continuous duality transformations as analysed here and the conventional one was derived for the Maxwell theory. Moreover it should be stressed that the duality symmetry (both $SO(2)$ and Z_2) in the Fourier modes holds for any model, irrespective of dimensionality, provided the system represents an assembly of decoupled (complex) harmonic oscillators. Then, for example, a scalar field in four dimensions, which does not display any duality symmetry in the configuration space, will still exhibit an analogous symmetry in the momentum space. We feel that this analysis of duality symmetry through a mode expansion can be pursued for other examples.

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